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Flat principal bundles over an abelian variety

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Abstract

We prove that a principal G-bundle E_G over a complex abelian variety A, where G is a complex reductive algebraic group, admits a flat holomorphic connection if and only if E_G is isomorphic to all the translations of it by the group structure of A.

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1. Introduction

Let G be a complex connected reductive linear algebraic group and A a complex abelian variety. A principal G-bundle E_G over A admits a flat holomorphic connection if E_G is given by a representation of the fundamental group $\pi_1(A)$ in G. If E_G admits a flat holomorphic connection then for any $x \in A$ the pullback $\tau_x^* E_G$ is isomorphic to E_G, where τ_x is the isomorphism of A defined by $y \mapsto x + y$. This can be proved using the fact that τ_x is homotopic to the identity automorphism of A.

We prove that the converse is true, namely, the condition that $\tau_x^* E_G$ is isomorphic to E_G for each $x \in A$ implies that E_G admits a flat holomorphic connection (Theorem 3.1). The proof of the theorem involves showing that such a translation invariant G-bundle is semistable.

We also show that E_G admits a flat holomorphic connection if it admits a holomorphic connection. In other words, the following three conditions are equivalent: (1) the isomorphism class of E_G is left invariant by translations in A; (2) E_G admits a holomorphic connection; (3) E_G admits a flat holomorphic connection.

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2. Preliminaries

Let A be an abelian variety over $\mathbb{C}$ of dimension d, with $d \geq 1$. Fix an ample line bundle L over A. For a coherent sheaf F on A, the degree of F is defined as

$$\text{degree}(F) := \int_A c_1(F)c_1(L)^{d-1}.$$

A torsionfree sheaf E on A is called *semistable* if for any nonzero coherent subsheaf F of E the inequality

$$\frac{\text{degree}(F)}{\text{rank}(F)} \leq \frac{\text{degree}(E)}{\text{rank}(E)}$$

is valid [5, p. 168]. For a torsionfree coherent sheaf F defined on a Zariski open subset $U \subset A$ with the codimension of the complement $A \setminus U$ at least two, the direct image $\iota_* F$ is a coherent sheaf on A, where ι is the inclusion map of U in A. For such a sheaf F define

$$\text{degree}(F) := \text{degree}(\iota_* F).$$

Let G be a complex connected reductive linear algebraic group. A principal G-bundle over A is a smooth complex variety E_G equipped with a right algebraic action of G and a smooth surjective morphism

$$p : E_G \to A, \tag{2.1}$$

such that

(1) the map p commutes with the actions of G, with G acting trivially on A;
(2) the map

$$E_G \times G \to E_G \times_A E_G,$$

to the fiber product defined by $(z, g) \mapsto (z, zg)$ is an isomorphism.

Ramanathan [7,8] extended the notion of semistability to G-bundles. We will briefly recall the definition. Take a principal G-bundle E_G over A. Let

$$E_P \subset E_G|_U,$$

be a reduction of structure group of E to a parabolic subgroup $P \subset G$ over a nonempty Zariski open subset $U \subset A$ with the codimension of the complement $A \setminus U$ being at least two. Let χ be a character of P dominant with respect to a Borel subgroup contained in P. The group P acts on $E_P \times \mathbb{C}$ as follows: the action of any $g \in P$ sends (z, c) to $(zg, \chi(g^{-1})c)$. So

$$E_\chi := \frac{E_P \times \mathbb{C}}{P}$$

is a line bundle over U, which is called the line bundle associated to E_P for χ.

The G-bundle E_G is called *semistable* if in *every* such situation describe above, the degree of the line bundle E_χ is nonpositive.

For any analytic open subset $U \subset A$, consider the inverse image $p^{-1}(U)$, where p is the projection in (2.1), which is a complex manifold equipped with an action of G. Let $\mathcal{C}(U)$ denote the space of holomorphic vector fields on $p^{-1}(U)$ that are left invariant by the action of G on $p^{-1}(U)$. Note that $\mathcal{C}(U)$ is closed under multiplication by holomorphic functions pulled back from U. Since the action of G is transitive on the fibers of p, the sheaf on A that associates to any U the vector space $\mathcal{C}(U)$ is coherent analytic. Let $\mathrm{At}(E_G)$ denote the corresponding holomorphic, hence algebraic, vector bundle over A defined by this coherent sheaf.

The vector bundle $\mathrm{At}(E_G)$ defined above is known as the *Atiyah bundle* for E_G [1]. Since $\mathcal{C}(U)$ is closed under the Lie bracket operation, there is an induced Lie algebra structure on the sections of $\mathrm{At}(E_G)$.

Let $\mathfrak{g}$ be the Lie algebra of G, and

$$\mathrm{ad}(E_G) := \frac{E_G \times \mathfrak{g}}{G},$$

the adjoint bundle; G acts on $\mathfrak{g}$ by conjugation. There is an exact sequence of vector bundles

$$0 \to \mathrm{ad}(E_G) \to \mathrm{At}(E_G) \xrightarrow{\sigma} TA \to 0, \tag{2.2}$$

over A where σ is defined using the differential $\mathrm{d}p$, and the subbundle $\mathrm{ad}(E_G)$ corresponds to the G-invariant vertical vector fields [1]. This sequence is known as the *Atiyah exact sequence*.

A *holomorphic connection* on E is a splitting of the Atiyah exact sequence, that is, a homomorphism of holomorphic vector bundles

$$D : TA \to \mathrm{At}(E_G),$$

such that $\sigma \circ D$ is the identity automorphism of TA [1]. The obstruction for D to be a Lie algebra homomorphism is the *curvature* of D. The curvature is a $\mathrm{ad}(E_G)$-valued holomorphic two form on A. The holomorphic connection D is called *flat* if its curvature vanishes identically.

A holomorphic connection on a vector bundle V is a first order holomorphic differential operator from V to $\Omega^1_A \otimes V$ satisfying the Leibniz identity. Note that a holomorphic connection on a holomorphic vector bundle of rank n is a holomorphic connection on the corresponding principal $\mathrm{GL}(n, \mathbb{C})$-bundle.

3. Translation invariant bundles

For any $x \in A$, let

$$\tau_x : A \to A,$$

be the translation map defined using the group structure of A. In other words, $\tau_x(y) = x + y$.

Theorem 3.1. *Let E_G be a principal G-bundle over A. The following three are equivalent*:

(1) *The G-bundle E_G admits a holomorphic connection.*

(2) *For each* $x \in X$, *the pullback* G-*bundle* $\tau_x^* E_G$ *is isomorphic to* E_G.
(3) *The* G-*bundle* E_G *admits a flat holomorphic connection.*

Proof. It is easy to see that (3) implies (2), as τ_x is homotopic to the identity map. To prove this assertion in detail, let D be a flat holomorphic connection on E_G. Fix a smooth path $\gamma : [0, 1] \to A$ connecting x with the identity element e of A. So, $\gamma(0) = e$ and $\gamma(1) = x$. Let

$$T : E_G \to \tau_x^* E_G,$$

be the isomorphism of G-bundles that sends the fiber of E over any $y \in A$ to the fiber of E over $x + y$ using parallel transport (for the connection D) along the path $\gamma_y : [0, 1] \to A$ defined by $\gamma_y(t) = \gamma(t) + y$. Since the holomorphic connection D is flat, the map T defined this way is a holomorphic isomorphism of G-bundles. Therefore, (3) implies (2).

We will now show that if E_G admits a holomorphic connection, then it admits a flat holomorphic connection. We will first recall from [2] a criterion for the existence of a flat holomorphic connection.

The G-bundle E_G admits a flat holomorphic connection if the following three conditions hold:

(a) the G-bundle E_G is semistable;
(b) $c_2(\mathrm{ad}(E_G)) \in H^4(A, \mathbb{Q})$ vanishes;
(c) for every character χ of G, the associated line bundle

$$E_\chi := \frac{E_G \times \mathbb{C}}{G}, \tag{3.1}$$

over A is of degree zero; the quotient is for the action of G defined as follows: the action of any $g \in G$ sends (z, c) to $(zg, \chi(g^{-1})c)$.

See [2, p. 205, Theorem 4.5].

Let D be a holomorphic connection on the G-bundle E_G. The holomorphic connection D induces a holomorphic connection on any vector bundle over A associated to E_G by some representation of G. In particular, both $\mathrm{ad}(E_G)$ and E_χ have induced holomorphic connections. Now, a theorem of Atiyah says that for a holomorphic vector bundle V with a holomorphic connection all the rational Chern classes of V (of positive degree) vanish [1, Theorem 4, p. 192].

Therefore, in view of the above criterion of [2], to prove (1) implies (3) in the statement of the theorem all we need to show is that E_G is semistable. Now, the G-bundle E_G is semistable if the vector bundle $\mathrm{ad}(E_G)$ is semistable [2, Lemma 4.3, p. 202]. (Actually the content of Lemma 4.3 of [2] is that E_G is semistable if and only if $\mathrm{ad}(E_G)$ is semistable. The assertion that E_G is semistable if $\mathrm{ad}(E_G)$ is semistable is actually a straight-forward consequence of the definition of semistability of E_G.) Since the holomorphic connection D on E_G induces a holomorphic connection on the vector bundle $\mathrm{ad}(E_G)$, to prove semistability of $\mathrm{ad}(E_G)$ it suffices to show that any vector bundle over A with a holomorphic connection is semistable.

Let F be a holomorphic vector bundle over A equipped with a holomorphic connection D_0. Consider the Harder–Narasimhan filtration of F [5, Theorem 7.15, p. 174]. If F is not semistable, let F_1 be the maximal semistable subsheaf of F, that is, F_1 is the first term in the Harder–Narasimhan filtration of F.

Let

$$S : F_1 \to \Omega_A^1 \otimes \left(\frac{F}{F_1}\right), \tag{3.2}$$

be the second fundamental form of F_1 for the connection D_0 on F. So S is the composition

$$F_1 \hookrightarrow F \xrightarrow{D_0} \Omega_A^1 \otimes F \xrightarrow{\mathrm{Id}\otimes q} \Omega_A^1 \otimes \left(\frac{F}{F_1}\right),$$

of homomorphisms of sheaves, where $q : F \to F/F_1$ is the quotient map. Note that the composition is $\mathcal{O}_A$-linear.

From the properties of the Harder–Narasimhan filtration it follows immediately that

$$H^0\left(A, \mathrm{Hom}\left(F_1, \frac{F}{F_1}\right)\right) = 0. \tag{3.3}$$

Indeed, the slope (=degree/rank) of F_1 is strictly greater than the slope of any coherent subsheaf of F/F_1. Since F_1 is semistable, any quotient of F_1 has slope at least that of F_1. Therefore, if there is a nonzero homomorphism of F_1 to F/F_1, then the image of the homomorphism contradicts the condition that the slope of F_1 is strictly greater than the slope of any subsheaf of F/F_1.

Since Ω_A^1 is the trivial vector bundle of rank d, we have

$$H^0\left(A, \mathrm{Hom}\left(F_1, \Omega_A^1 \otimes \left(\frac{F}{F_1}\right)\right)\right) = H^0\left(A, \mathrm{Hom}\left(F_1, \frac{F}{F_1}\right)\right) \otimes_{\mathbb{C}} \mathbb{C}^d.$$

Now (3.3) gives

$$H^0\left(A, \mathrm{Hom}\left(F_1, \Omega_A^1 \otimes \left(\frac{F}{F_1}\right)\right)\right) = 0.$$

In particular, $S = 0$, where S is defined in (3.2). In other words, the subsheaf F_1 of F is preserved by D. Therefore, degree$(F_1) = 0$ [1, Theorem 4, p. 192]. (Note that since F_1 is torsionfree, it is locally free on a Zariski open subset U with the codimension of $A \setminus U$ at least two. Therefore, degree(F_1) = degree$(F_1|_U)$.) On the other hand degree$(F) = 0$ as F admits a holomorphic connection. Since degree(F_1) = degree(F), the vector bundle F must be semistable. Consequently, F admits a flat holomorphic connection by the criterion of [2].

Now we will show that (2) implies (3) in the statement of the theorem. So, let E_G be a principal G-bundle over A with the property that $\tau_x^* E_G$ is isomorphic to E_G for each $x \in A$.

Let V be a complex left G-module of dimension n. Consider the associated vector bundle

$$E_V := \frac{E_G \times V}{G},$$

of rank n over A, where the quotient is for the action of G defined as follows: the action of any $g \in G$ sends $(z, v) \in E_G \times V$ to $(zg, g^{-1}v)$. The condition that $\tau_x^* E_G$ is isomorphic to

E_G implies that the vector bundle $\tau_x^* E_V$ is isomorphic to E_V. In particular, the line bundle

$$\mathcal{L} := \bigwedge^n E_V$$

(the dimension of V is n) over A has the property that $\tau_x^* \mathcal{L}$ is isomorphic to $\mathcal{L}$ for each $x \in A$. This implies that

$$\text{degree}(\mathcal{L}) = 0 \tag{3.4}$$

In [6, Definition, p. 74], Mumford defines $\text{Pic}^0(A)$ to be the group of all line bundles L' over A with the property that $\tau_x^* L'$ is isomorphic to L' for each $x \in A$ (the map ψ in [6, Definition, p. 74] is defined in p. 60 of [6]). Then it is shown that $\text{Pic}^0(A)$ (with this definition) coincides with the usual $\text{Pic}^0(A)$, namely, the group of topologically trivial holomorphic line bundles over A (see [6, p. 86]).

If the vector bundle E_V is not semistable, let

$$V' \subset E_V,$$

be the maximal semistable subsheaf of E_V, that is, the subsheaf V' is the first term in the Harder–Narasimhan filtration of E_V. Since

$$\tau_x^* E_V \cong E_V,$$

from the uniqueness of the Harder–Narasimhan filtration it follows immediately that

$$\tau_x^* V' \cong V'.$$

Therefore, the determinant line bundle $\wedge^{\text{top}} V'$ has the property that

$$\tau_x^* \wedge^{\text{top}} V' \cong \wedge^{\text{top}} V'.$$

As we saw in (3.4), this implies that $\text{degree}(V') = \text{degree}(\wedge^{\text{top}} V') = 0$. Since

$$\text{degree}(E_V) = 0 = \text{degree}(V'),$$

we conclude that E_V is semistable.

Consequently, the condition that the pullback G-bundle $\tau_x^* E_G$ is isomorphic to E_G for any $x \in A$ implies that the vector bundle $\text{ad}(E_G)$ is semistable and the line bundle E_χ in (3.1) is of degree zero. In view of the criterion of [2] for the existence of a flat holomorphic connection (criterion described earlier), to complete the proof we need to show that $c_2(\text{ad}(E_G)) \in H^4(A, \mathbb{Q})$ vanishes.

Let

$$C_2(\text{ad}(E_G)) \in \text{CH}^2(A), \tag{3.5}$$

be the image of the second Chern class of the vector bundle $\text{ad}(E_G)$ in the Chow group (see Chapter 3 of [4]). The condition that

$$\tau_x^* \,\text{ad}(E_G) \cong \text{ad}(E_G)$$

for each $x \in A$ implies that the element $C_2(\text{ad}(E_G)) \in \text{CH}^2(A)$ is left invariant by the action of A on $\text{CH}^2(A)$ by translations.

We will now show that if an element $c \in \mathrm{CH}^2(A)$ has the property that

$$c = \tau_x(c) \in \mathrm{CH}^2(A)$$

for each $x \in X$, then the cycle class of c in $H^4(A, \mathbb{Q})$ vanishes.

The cycle class map will be denoted by ψ. Since τ_x is homotopic to the identity map of A, we have

$$\psi(\tau_x(c)) = \psi(c) \in H^4(A, \mathbb{Q}),$$

that is, $\tau_x(c)$ and c are homologically equivalent. So the image of $\tau_x(c) - c \in \mathrm{CH}^2(A)$ by the Abel–Jacobi map AJ_A

$$\mathrm{AJ}_A(\tau_x(c) - c) \in J^2(A) := \frac{H^3(A, \mathbb{C})}{F^2H^3(A, \mathbb{C}) + H^3(A, \mathbb{Z})}.$$

See [3, p. 22] for the Abel–Jacobi map and the definition of $J^2(A)$. Let

$$I : A \to J^2(A), \tag{3.6}$$

be the map that sends any $y \in A$ to $\mathrm{AJ}_A(\tau_y(c) - c) \in J^2(A)$. This map I is holomorphic, which is a consequence of the properties of $J^2(A)$.

The holomorphic cotangent space Ω^1_e of A at the identity element e will be denoted by W. So $H^{i,j}(A)$ is naturally identified with $(\wedge^i W) \otimes (\wedge^j \bar{W})$, where $\bar{W}$ is the conjugate of W, equivalently, $\bar{W} = (T^{0,1}_e A)^*$.

Consider the differential

$$\mathrm{d}I(e) : W^* \to T_0J^2(A) = \wedge^3\bar{W} \oplus (W \otimes \wedge^2\bar{W}), \tag{3.7}$$

at the point $e \in A$ of the map I constructed in (3.6); here $0 \in J^2(A)$ is the identity element. Note that

$$\psi(c) \in H^{2,2}(A) = \wedge^2 W \otimes \wedge^2\bar{W}.$$

The homomorphism $\mathrm{d}I(e)$ in (3.7) is the contraction of $\psi(c)$. In other words

$$\mathrm{d}I(e)(w) = \langle w, \psi(c)\rangle \in W \otimes \wedge^2\bar{W} \subset T_0J^2(A)$$

for all $w \in W^*$, where $\langle -, -\rangle$ is the contraction of W^* with W. That $\mathrm{d}I(e)$ is the contraction homomorphism is a straight-forward consequence of the description of the differential of the Abel–Jacobi map (see [3, p. 28]).

Therefore, the condition

$$\tau_x(c) = c \in \mathrm{CH}^2(A)$$

for all $x \in A$ implies that $\psi(c) = 0$ (as the contraction with W^* vanishes identically).

Consequently, $\psi(C_2(\mathrm{ad}(E_G))) = 0$. But $\psi(C_2(\mathrm{ad}(E_G))) = c_2(\mathrm{ad}(E_G)) \in H^4(A, \mathbb{Q})$. So E_G admits a flat holomorphic connection by the criterion of [2]. This completes the proof of the theorem. □

Let $x' \in A$ be such that the subgroup of A generated by x' is Zariski dense in A. It is easy to see that such an element exists. In fact, the subset defined by all elements of A with this property is Zariski dense.

Theorem 3.1 has the following corollary.

Corollary 3.2. *Let E_G be a principal G-bundle over A such that $\tau^*_{x'}E_G$ is isomorphic to E_G. Then E_G admits a flat holomorphic connection.*

Proof. For any character χ of G, consider the line bundle E_χ defined in (3.1). Let $c := c_1(E_\chi) \in H^2(A, \mathbb{Q})$ be the first Chern class, and denote by χ' the element defined by E_χ in the component $\mathrm{Pic}^c(A)$ of the Picard group. Let $H_{x'} \subset A$ be the subgroup generated by x'. Since $\tau(\chi') = \chi'$ for all $x \in H_{x'}$ and $H_{x'}$ is Zariski dense in A, we have $\tau(\chi') = \chi'$ for all $x \in A$. As we saw in the proof of Theorem 3.1, this implies that $c_1(E_\chi) = [\chi'] = 0$.

Similarly, since $C_2(\mathrm{ad}(E_G))$ (defined in (3.5)) coincides with $\tau_x(C_2(\mathrm{ad}(E_G)))$ for all x in the Zariski dense subgroup $H_{x'}$, we have $\tau_x(C_2(\mathrm{ad}(E_G))) = C_2(\mathrm{ad}(E_G))$ for all $x \in A$. Consequently, as in the proof of Theorem 3.1, we conclude that $c_2(\mathrm{ad}(E_G)) = 0$.

In the proof of Theorem 3.1 we saw that if E_G is isomorphic to all translations of it, then any associated vector bundle is semistable. This proof clearly goes through under the assumption that $\tau^*_y E_G \cong E_G$ for all y in the Zariski dense subgroup $H_{x'}$. Therefore, the proof of the corollary is complete. □

The above corollary can be reformulated as follows.

Let $x_0 \in A$ and E_G a G-bundle such that $\tau_{x_0} E_G \cong E_G$. Let $A_{x_0} \subset A$ be the connected component of the Zariski closure of the subgroup generated by x_0 containing the identity element. So A_{x_0} is a subabelian variety of A.

Corollary 3.3. *The G-bundle $E_G|_{A_{x_0}}$ over the abelian variety A_{x_0} admits a flat holomorphic connection.*

This corollary follows from Corollary 3.2 by setting $A = A_{x_0}$.

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